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Journal of Sound and Vibration 306 (2007) 675–690
JOURNAL OF SOUND AND VIBRATION
www.elsevier.com/locate/jsvi

On the boundary conditions for axially moving beams

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Received 11 August 2006; received in revised form 3 June 2007; accepted 7 June 2007

Abstract

Axially moving beam-typed structures are of technical importance and present in a wide class of engineering problem. As the axial speed of a beam may significantly affect the dynamic characteristics of the structure even at a low velocity, it is important to accurately predict the dynamic characteristics and stability of such structures. In most previous studies, the net energy flux through the left-end and right-end boundaries of the finite beam over two simple supports has been implicitly assumed to be zero by completely ignoring the effects of its left (incoming) and right (outgoing) semi-infinite beam parts or by applying fixed boundary conditions for its longitudinal vibration, which seems to be very non-realistic from the physical point of view. Thus, this paper investigates the effects of the continuously incoming and outgoing semi-infinite beam parts on the dynamic characteristics and stability of an axially moving beam by using the spectral element method. The spectral element model is formulated from the equations of motion derived by using the Hamilton's principle extended for the systems of changing mass. It is numerically shown that the effects of the continuously incoming and outgoing semi-infinite beam parts should be taken into account for further accurate prediction of the dynamic characteristics and stability for such axially moving beams.

1. Introduction

The axially moving structures have received a great deal of attention due to their manifestation in a wide class of engineering fields. The belt drives, high-speed magnetic tapes, band saw blades, fiber winding, and the aerial cable tramways are the typical examples of such axially moving structures. Many researchers have shown that the axial speed of a structure plays an important role on its vibration characteristics and dynamic stability. Above a certain critical axial speed, the structure may lose its dynamic stability to result in a severe vibration. Thus, it is important to accurately predict the axial speed-dependent dynamic characteristics and instability for the successful design and safe operation of such structures. Recent developments in research on axially moving structures have been reviewed in Refs. [1,2].

Most of the one-dimensional structures with flexural rigidity, which are axially moving over two simple supports, have been represented by the finite beam models [2–32] and the vibration analyses have been conducted by using various solution techniques such as the Galerkin's method [2–9], assumed mode method [10], finite element method (FEM) [11,12], Green's function method [13], transfer function method [14],

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0022-460X/$ - see front matter
doi:10.1016/j.jsv.2007.06.039

perturbation method [15–17], asymptotic method [18,19], Laplace transform method [28], and spectral element method (SEM) [29–32]. The SEM is known as an exact frequency-domain solution method, in which the spectral element matrix (often called exact dynamic stiffness matrix) is used for the dynamic analysis. As an obvious advantage over the classic FEM, a whole beam without any material or geometric discontinuities can be treated as a single finite element, regardless of its length, which still provides exact solutions. In SEM, the dynamic responses in the frequency and time domains are computed very efficiently by using the FFT algorithm. For further details of SEM, the readers are referred to Refs. [33–35].

In most previous studies [2–31], the dynamic characteristics and stabilities of axially moving beams have been investigated only for the spatial domain of problem defined by the finite span of beam between two simple supports, for instance, by ignoring the semi-infinite beam parts which are continuously moving into and moving away from the spatial domain of the problem. Traditionally most researchers have implicitly assumed that the net energy flux through the left and right boundaries of the finite span of beam is zero and have applied fixed boundary conditions for the longitudinal (axial) vibration (e.g., Refs. [2,18,32]). From the physical point of view, ignoring the continuously incoming and outgoing semi-infinite beam parts, assuming zero net energy flux through the left-end and right-end boundaries of the finite span of beam, or applying the fixed boundary conditions for the longitudinal vibration seems to be quite non-realistic.

In the present paper, we investigate the effects of the continuously incoming and outgoing semi-infinite beam parts on the dynamic characteristics and stability of an axially moving uniform beam by using the SEM. To this end, the finite span of beam between two simple supports is represented by the two-noded spectral element model and the incoming and outgoing semi-infinite beam parts by the semi-infinite spectral element models, without applying the fixed boundary conditions for the longitudinal vibration.

2. Governing equations

Consider a uniform beam is moving in its axial (x) direction at a constant moving speed of v between two simple supports as shown in Fig. 1. The span between two simple supports is l. The structural properties of the beam are given by the axial (extensional) rigidity EA, the flexural rigidity EI, and the mass density per unit length of the beam ρA. Assume that the beam has small amplitude vibrations in the axial and transverse directions and neglect the coupling effects between the axial displacement and the transverse displacement for the brevity without impeding the main purpose of this study.

Because of the continuing mass transport across the boundaries at $x = 0$ and l, an extended form of Hamilton's principle developed by McIver [36] for the systems of changing mass is used to derive the governing equations of motion. The extended form of Hamilton's principle is given by

$$\int_{t_1}^{t_2} (\delta K - \delta P + \delta W_{\mathrm{NC}} + \delta W_{\text{mass transport}})\mathrm{d}t = 0, \tag{1}$$

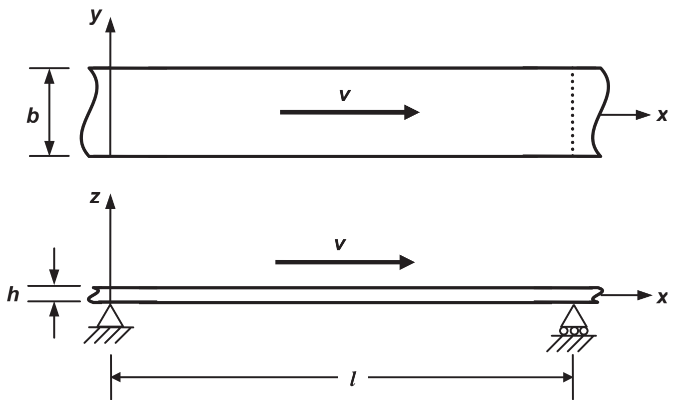

Fig. 1. Geometry of an axially moving uniform beam.

where K is the kinetic energy, P the strain energy, and $\mathrm{d}W_{\mathrm{NC}}$ the virtual work done by non-conservative forces $p_x(x,t)$ and $p_z(x,t)$ externally applied in the axial and transverse directions, respectively. They are given by

$$K = \frac{1}{2}\int_0^l \{\rho A(v + \dot{u} + vu')^2 + \rho A(\dot{w} + vw')^2 + \rho I \dot{w}'^2\}\,\mathrm{d}x,$$
$$P = \frac{1}{2}\int_0^l (EIw''^2 + EAu'^2)\,\mathrm{d}x,$$
$$\delta W_{\mathrm{NC}} = \int_0^l (p_x\,\delta u + p_z\,\delta w)\,\mathrm{d}x. \tag{2}$$

The prime and dot denote the derivatives with respective to the spatial coordinate x and the time t, respectively. The kinetic energy K consists of the energies due to the translational motions in the x- and z-directions (first and second terms) and the rotational motion (third term), and the strain energy P consists of the energies due to the bending (first term) and the axial deformation (second term). The last term $\mathrm{d}W_{\text{mass transport}}$ of Eq. (1) is the virtual momentum transport of mass across the boundaries at $x = 0$ and l and it is given by [36–38]

$$\delta W_{\text{mass transport}} = \rho A v(\mathbf{v}\cdot\delta\mathbf{r})\big|_0^l, \tag{3}$$

where $\mathbf{r}$ is the position vector of a point at x, $\delta\mathbf{r}$ the virtual displacement, and $\mathbf{v} = \mathrm{d}\mathbf{r}/\mathrm{d}t$ the velocity field of the moving beam at time t. They are given by

$$\mathbf{r} = (x+u)\mathbf{n}_x + w\mathbf{n}_z, \quad \delta\mathbf{r} = \delta u\mathbf{n}_x + \delta w\mathbf{n}_z,$$
$$\mathbf{v} = (v + \dot{u} + vu')\mathbf{n}_x + (\dot{w} + vw')\mathbf{n}_z, \tag{4}$$

where $\mathbf{n}_x$ and $\mathbf{n}_z$ are the unit direction vectors in the x- and z-directions, respectively. By substituting Eq. (4) into Eq. (3), we get

$$\delta W_{\text{mass transport}} = [\rho A v(v + \dot{u} + vu')\delta u + \rho A v(\dot{w} + vw')]|_0^l. \tag{5}$$

By substituting Eqs. (2) and (5) into Eq. (1), the equations of motion can be derived as

$$\begin{aligned} EAu'' - 2\rho A v\dot{u}' - \rho A v^2 u'' - \rho A\ddot{u} &= -p_x(x,t),\\ EIw'''' + \rho A v^2 w'' + 2\rho A v\dot{w}' - \rho I\ddot{w}'' + \rho A\ddot{w} &= p_z(x,t), \end{aligned} \tag{6}$$

together with the relevant force–displacement relations as

$$M(x,t) = EIw'',$$
$$V(x,t) = -EIw''' + \rho I\ddot{w}',$$
$$N(x,t) = EAu', \tag{7}$$

where M is the bending moment, V the transverse shear force, and N the axial tensile force.

3. Spectral elements formulation

Based on the DFT theory [39], assume the displacement fields in the spectral forms as

$$u(x,t) = \sum_{n=0}^{N-1} U_n(x)\,\mathrm{e}^{\mathrm{i}\omega_n t}, \quad w(x,t) = \sum_{n=0}^{N-1} W_n(x)\,\mathrm{e}^{\mathrm{i}\omega_n t} \tag{8}$$

where $U_n(x)$ and $W_n(x)$ represent the spectral (Fourier) components of $u(x,\ t)$ and $w(x,\ t)$, respectively. Similarly, we express the external loads into the spectral forms as

$$p_x(x,t) = \sum_{n=0}^{N-1} P_{xn}(x)\,\mathrm{e}^{\mathrm{i}\omega_n t}, \quad p_z(x,t) = \sum_{n=0}^{N-1} P_{zn}(x)\,\mathrm{e}^{\mathrm{i}\omega_n t}, \tag{9}$$

where $P_{xn}(x)$ and $P_{zn}(x)$ are the spectral components of $p_x(x,t)$ and $p_z(x,t)$, respectively. Substituting Eqs. (8) and (9) into Eqs. (6) and (9), we get

$$(EA - \rho A v^2)U_n'' - 2\mathrm{i}\omega_n \rho A v U_n' + \rho A \omega_n^2 U_n = -P_{xn},$$
$$EIW_n'''' + (\rho A v^2 + \rho I \omega_n^2)W_n'' + 2\mathrm{i}\rho A v \omega_n W_n' - \rho A \omega_n^2 W_n = P_{zn} \tag{10}$$

and

$$N_n(x) = EAU_n',$$
$$V_n(x) = -EIW_n''' - \rho I \omega_n^2 W_n',$$
$$M_n(x) = EIW_n'', \tag{11}$$

where $N_n(x)$, $V_n(x)$, and $M_n(x)$ are the spectral components of $N(x,t)$, $V(x,t)$, and $M(x,t)$, respectively.

As the first step of spectral element formulation [33–35], we consider the homogeneous governing equations reduced from Eq. (10) as

$$(EA - \rho A v^2)U_n'' - 2\mathrm{i}\omega_n \rho A v U_n' + \rho A \omega_n^2 U_n = 0,$$
$$EIW_n'''' + (\rho A v^2 + \rho I \omega_n^2)W_n'' + 2\mathrm{i}\omega_n \rho A v W_n' - \rho A \omega_n^2 W_n = 0. \tag{12}$$

The general solutions of Eq. (12) can be assumed as

$$U_n(x) = \sum_{i=1}^{2} A_{ni}\, \mathrm{e}^{\mathrm{i}k_{ni}x}, \quad W_n(x) = \sum_{j=1}^{4} B_{nj}\, \mathrm{e}^{\mathrm{i}\lambda_{nj}x} \tag{13}$$

or

$$U_n(x) = [\mathbf{E}_U(x;\omega_n)]\{\mathbf{C}_n\},$$
$$W_n(x) = [\mathbf{E}_W(x;\omega_n)]\{\mathbf{C}_n\}, \tag{14}$$

where

$$[\mathbf{E}_U(x;\omega_n)] = [\,\mathrm{e}^{\mathrm{i}k_{n1}x} \quad 0 \quad 0 \quad \mathrm{e}^{\mathrm{i}k_{n2}x} \quad 0 \quad 0\,].$$
$$[\mathbf{E}_W(x;\omega_n)] = [\,0 \quad \mathrm{e}^{\mathrm{i}\lambda_{n1}x} \quad \mathrm{e}^{\mathrm{i}\lambda_{n2}x} \quad 0 \quad \mathrm{e}^{\mathrm{i}\lambda_{n3}x} \quad \mathrm{e}^{\mathrm{i}\lambda_{n4}x}\,].$$
$$\{\mathbf{C}_n\} = \{\,A_{n1} \quad B_{n1} \quad B_{n2} \quad A_{n2} \quad B_{n3} \quad B_{n4}\,\}^{\mathrm{T}}. \tag{15}$$

In above equations, k_{ni} $(i = 1,2)$ and λ_{nj} $(j = 1,2,3,4)$ are the wavenumbers for the longitudinal (axial) and bending wave modes, respectively, and they are computed from the dispersion relations given by

$$-(EA - \rho A v^2)k_n^2 + 2\omega_n \rho A v k_n + \rho A \omega_n^2 = 0,$$
$$EI\lambda_n^4 - (\rho A v^2 + \rho I \omega_n^2)\lambda_n^2 - 2\omega_n \rho A v \lambda_n - \rho A \omega_n^2 = 0. \tag{16}$$

The constant vector $\{\mathbf{C}_n\}$ should be determined to satisfy the boundary conditions for the problem under consideration.

3.1. Spectral element for the finite beam

Consider a finite beam of length l as shown in Fig. 2. The spectral components of the nodal degrees of freedom (simply, spectral nodal dofs) shown in Fig. 2 are defined by

$$U_{n1} = U_{n1}(0), \quad W_{n1} = W_{n1}(0), \quad \Phi_{n1} = W_{n1}'(0),$$
$$U_{n2} = U_{n2}(l), \quad W_{n2} = W_{n2}(l), \quad \Phi_{n2} = W_{n2}'(l). \tag{17}$$

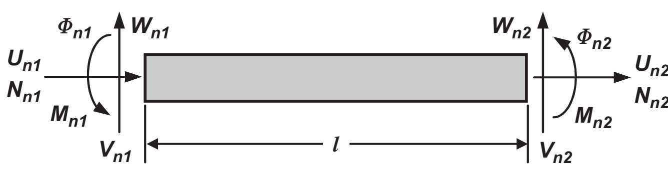

Fig. 2. Sign convention for the spectral element model of a finite beam.

By substituting the above the boundary conditions into Eq. (14), we first obtain the constant vector $\{\mathbf{C}_n\}$ in terms of the spectral nodal dofs vector $\{\mathbf{d}_n\}$ and use the result to express the displacement fields as

$$U_n(x) = [\mathbf{E}_U][\mathbf{X}_n]^{-1}\{\mathbf{d}_n\} \equiv [\mathbf{N}_U(x;\omega_n)]\{\mathbf{d}_n\},$$
$$W_n(x) = [\mathbf{E}_W][\mathbf{X}_n]^{-1}\{\mathbf{d}_n\} \equiv [\mathbf{N}_W(x;\omega_n)]\{\mathbf{d}_n\}, \tag{18}$$

where

$$\{\mathbf{d}_n\} = \{\, U_{n1} \quad W_{n1} \quad \Phi_{n1} \quad U_{n2} \quad W_{n2} \quad \Phi_{n2}\,\}^{\mathrm{T}},$$

$$[\mathbf{X}_n] = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & \mathrm{i}\lambda_{n1} & \mathrm{i}\lambda_{n2} & 0 & \mathrm{i}\lambda_{n3} & \mathrm{i}\lambda_{n4} \\ \mathrm{e}^{\mathrm{i}k_{n1}l} & 0 & 0 & \mathrm{e}^{\mathrm{i}k_{n2}l} & 0 & 0 \\ 0 & \mathrm{e}^{\mathrm{i}\lambda_{n1}l} & \mathrm{e}^{\mathrm{i}\lambda_{n2}l} & 0 & \mathrm{e}^{\mathrm{i}\lambda_{n3}l} & \mathrm{e}^{\mathrm{i}\lambda_{n4}l} \\ 0 & \mathrm{i}\lambda_{n1}\,\mathrm{e}^{\mathrm{i}\lambda_{n1}l} & i\lambda_{n2}\,\mathrm{e}^{\mathrm{i}\lambda_{n2}l} & 0 & \mathrm{i}\lambda_{n3}\,\mathrm{e}^{\mathrm{i}\lambda_{n3}l} & \mathrm{i}\lambda_{n4}\,\mathrm{e}^{\mathrm{i}\lambda_{n4}l} \end{bmatrix}, \tag{19}$$

where $[\mathbf{N}_{\mathrm{Un}}]$ and $[\mathbf{N}_{\mathrm{Wn}}]$ are the frequency-dependent dynamic shape function matrices.

We use the variational approach [40] to formulate the spectral element matrix. To this end, we consider the weak form statements of the governing Eq. (10) as follows:

$$\int_0^l \{(EA - \rho A v^2)U_n'' - 2\mathrm{i}\omega_n \rho A v U_n' + \rho A \omega_n^2 U_n + P_{xn}\}\delta U_n \,\mathrm{d}x = 0,$$
$$\int_0^l \{EI W_n'''' + (\rho A v^2 + \rho I \omega_n^2) W_n'' + 2\mathrm{i}\omega_n \rho A v W_n' - \rho A \omega_n^2 W_n - P_{nz}\}\delta W_n \,\mathrm{d}x = 0. \tag{20}$$

By integrating Eq. (20) by parts and applying Eqs. (11) and (18), we can obtain a matrix equation as

$$[\mathbf{S}_n]\{\mathbf{d}_n\} = \{\mathbf{f}_n\}, \tag{21}$$

where $[\mathbf{S}_n]$ is the frequency-dependent spectral element matrix for the finite length of moving beam given by

$$\begin{aligned}[\mathbf{S}_n] = &\int_0^l \{(EA - \rho A v^2)[\mathbf{N}_{\mathrm{Un}}']^{\mathrm{T}}[\mathbf{N}_{\mathrm{Un}}'] - \rho A \omega_n^2 [\mathbf{N}_{\mathrm{Un}}]^{\mathrm{T}}[\mathbf{N}_{\mathrm{Un}}] + \mathrm{i}\rho A v \omega_n [[\mathbf{N}_{\mathrm{Un}}]^{\mathrm{T}}[\mathbf{N}_{\mathrm{Un}}'] - [\mathbf{N}_{\mathrm{Un}}']^{\mathrm{T}}[\mathbf{N}_{\mathrm{Un}}]]\} \\ &+ \int_0^l \{EI[\mathbf{N}_{\mathrm{Wn}}'']^{\mathrm{T}}[\mathbf{N}_{\mathrm{Wn}}''] - (\rho A v^2 + \rho I \omega_n^2)[\mathbf{N}_{\mathrm{Wn}}']^{\mathrm{T}}[\mathbf{N}_{\mathrm{Wn}}'] + \mathrm{i}\rho A v \omega_n ([\mathbf{N}_{\mathrm{Wn}}]^{\mathrm{T}}[\mathbf{N}_{\mathrm{Wn}}'] \\ &- [\mathbf{N}_{\mathrm{Wn}}']^{\mathrm{T}}[\mathbf{N}_{\mathrm{Wn}}]) - \rho A \omega_n^2 [\mathbf{N}_{\mathrm{Wn}}]^{\mathrm{T}}[\mathbf{N}_{\mathrm{Wn}}]\}\,\mathrm{d}x\end{aligned} \tag{22}$$

and the vector $\{\mathbf{f}_n\}$ is the spectral nodal forces vector given by

$$\begin{aligned}\{\mathbf{f}_n\} = &\{\, N_{1n} \quad 0 \quad 0 \quad N_{2n} \quad 0 \quad 0\,\}^{\mathrm{T}} + \int_0^l P_{xn}(x)[\mathbf{N}_{\mathrm{Un}}]^{\mathrm{T}}\,\mathrm{d}x \\ &+ \{\, 0 \quad V_{1n} \quad M_{1n} \quad 0 \quad V_{2n} \quad M_{2n}\,\}^{\mathrm{T}} + \int_0^l P_{zn}(x)[\mathbf{N}_{\mathrm{Wn}}]^{\mathrm{T}}\,\mathrm{d}x\end{aligned} \tag{23}$$

The further details of $[\mathbf{S}_n]$ are given in Appendix A.

3.2. Spectral elements for the semi-infinite beams

We need to consider two semi-infinite beams: one is on the left-hand side of the finite beam span between two simple supports and the other one is on the right-hand side of the finite beam span. The left semi-infinite beam is continuously moving into the spatial domain of problem, while the semi-infinite beam is moving out from the spatial domain. We need to determine the directions of each traveling waves (i.e., the right-moving waves or the left-moving waves) to choose correct waves needed to formulate the spectral elements for such semi-infinite beams.

For the case of longitudinal waves, it is quite easy and straightforward to determine the directions of the corresponding waves. The first dispersion relation of Eq. (16) always provides two real wavenumbers. The positive real wavenumber (denoted by k_{n1}) represents the left-moving wave mode and the negative real wavenumber (denoted by k_{n2}) represents the right-moving wave mode.

For the case of bending waves, the second dispersion relation of Eq. (16) provides total four wavenumbers, denoted by λ_{n1}, λ_{n2}, λ_{n3}, and λ_{n4}. The direction of a bending wave mode is determined by the sign of the real part of the corresponding wavenumber [41]: the positive sign means the left-moving wave mode and the negative sign the right-moving wave mode. The positive imaginary part of a wavenumber implies that the corresponding wave mode is the evanescent wave mode. The phase speed of the bending wave mode corresponding to the wavenumber λ_{nr} $(r = 1, 2, 3, 4)$, at a specific frequency ω_n, is defined by [41]

$$c_{nr} = \frac{\omega_n}{\lambda_{nr}} \quad (r = 1, 2, 3, 4). \tag{24}$$

Define two dimensionless speeds as follows:

$$\begin{aligned} &\text{Dimensionless phase speed}: \quad C_{nr} = \frac{c_{nr}}{c_{no}}, \\ &\text{Dimensionless axial speed}: \quad V_n = \frac{v}{c_{no}}. \end{aligned} \tag{25}$$

where c_{no} is the phase speed of the stationary beam (i.e., $v = 0\,\text{m/s}$) defined by

$$c_{no} = \left(\frac{EI}{\rho A} \omega_n^2 \right)^{1/4} \tag{26}$$

Fig. 3 shows the dimensionless phase speeds C_{nr} $(r = 1, 2, 3, 4)$ as the functions of the dimensionless moving speed V_n for each bending wave modes. Fig. 3 enables us to determine which wavenumbers (among four wavemumbers) correspond to the right-moving wave modes: then the remainder will correspond to the left-moving wave modes. In Fig. 3, C_{n1} is used to denote the evanescent wave in the left direction, C_{n2} the evanescent wave mode in the right direction, C_{n3} the right-moving wave mode, and C_{n4} the left-moving wave mode at zero moving speed. It is interesting to observe from Fig. 3 that, as the effect of the moving speed which is positive when a beam is moving in the right direction, two initially evanescent waves are all transformed into the right-moving wave modes when the moving speed becomes larger than the group speed which is the twice of the phase speed c_{no} [41].

3.2.1. *Spectral element for the left semi-infinite beam*

The spectral components of the displacement fields within the left semi-infinite beam shown in Fig. 4(a) are assumed in the forms as

$$\begin{aligned} {}^{L}U_n(x) &= [{}^{L}\mathbf{E}_U(x;\omega_n)]\{{}^{L}\mathbf{C}_n\}, \\ {}^{L}W_n(x) &= [{}^{L}\mathbf{E}_W(x;\omega_n)]\{{}^{L}\mathbf{C}_n\}, \end{aligned} \tag{27}$$

where

$$\begin{aligned} [{}^{L}\mathbf{E}_U(x;\omega_n)] &= [\,e^{i\kappa_{n1}x} \quad 0 \quad 0\,], \\ [{}^{L}\mathbf{E}_W(x;\omega_n)] &= [\,0 \quad e^{i\lambda_{n1}x} \quad e^{i\lambda_{n4}x}\,], \\ \{{}^{L}\mathbf{C}_n\} &= \{\,A_{n1} \quad B_{n1} \quad B_{n4}\,\}^{\mathrm{T}}. \end{aligned} \tag{28}$$

The superscript L denotes the quantities for the left semi-infinite moving beam. As shown in Fig. 4(a), the left semi-infinite moving beam has only one node at its right end. Thus we define the spectral nodal dofs for the one-noded spectral element model (often called 'throw-off element' model [33]) of the left semi-infinite moving

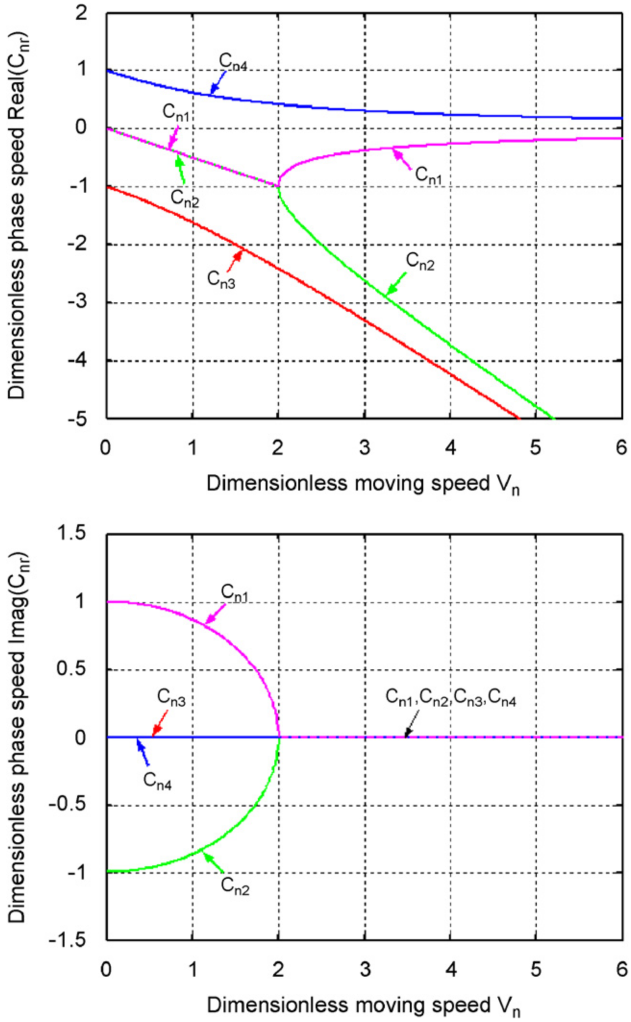

Fig. 3. Dimensionless phase speed C_{nr} vs. dimensionless axial speed V_n.

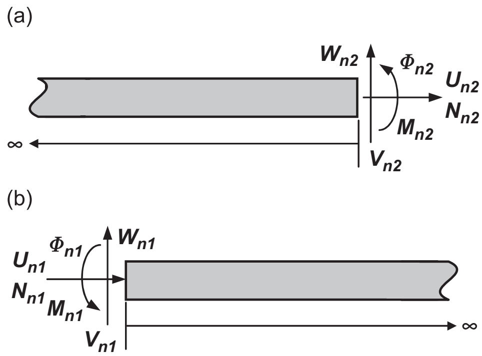

Fig. 4. Sign convention for the semi-infinite spectral element models: (a) left semi-infinite beam; (b) right semi-infinite beam.

beam as

$$\{{}^L\mathbf{d}_n\} = \begin{Bmatrix} U_{n1} \\ W_{n1} \\ \Phi_{n1} \end{Bmatrix} = \begin{Bmatrix} {}^LU_n(0) \\ {}^LW_n(0) \\ {}^LW'_n(0) \end{Bmatrix}. \tag{29}$$

By substituting Eq. (27) into Eq. (29), we obtain the constants vector $\{{}^L\mathbf{C}_n\}$ in terms of $\{{}^L\mathbf{d}_n\}$ and use the result to rewrite Eq. (27) as

$$\begin{aligned} {}^LU_n(x) &= [{}^L\mathbf{N}_U(x;\omega_n)]\{{}^L\mathbf{d}_n\}, \\ {}^LW_n(x) &= [{}^L\mathbf{N}_W(x;\omega_n)]\{{}^L\mathbf{d}_n\}, \end{aligned} \tag{30}$$

where

$$\begin{aligned} [{}^L\mathbf{N}_U(x;\omega_n)] &= [{}^L\mathbf{E}_U(x;\omega_n)][{}^L\mathbf{X}_n]^{-1}, \\ [{}^L\mathbf{N}_W(x;\omega_n)] &= [{}^L\mathbf{E}_W(x;\omega_n)][{}^L\mathbf{X}_n]^{-1}, \end{aligned} \tag{31}$$

with the use of definition:

$$[{}^L\mathbf{X}_n] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & \mathrm{i}\lambda_{n1} & \mathrm{i}\lambda_{n4} \end{bmatrix}. \tag{32}$$

Now define the spectral nodal forces and moments as

$$\{{}^L\mathbf{f}_n\} = \begin{Bmatrix} N_{n1} \\ V_{n1} \\ M_{n1} \end{Bmatrix} = \begin{Bmatrix} N_n(x=0) \\ V_n(x=0) \\ M_n(x=0) \end{Bmatrix}. \tag{33}$$

By substituting Eq. (30) into Eq. (11) and applying the results into Eq. (33), we can derive the relationship between $\{{}^L\mathbf{d}_n\}$ and $\{{}^L\mathbf{f}_n\}$ as

$$[{}^L\mathbf{S}_n]\{{}^L\mathbf{d}_n\} = \{{}^L\mathbf{f}_n\}, \tag{34}$$

where $[{}^L\mathbf{S}_n]$ is the spectral element matrix for the left semi-infinite moving beam given by

$$[{}^L\mathbf{S}_n] = \begin{bmatrix} EA[{}^L\mathbf{N}'_U(0;\omega_n)] \\ -EI[{}^L\mathbf{N}'''_W(0;\omega_n)] - \rho I\omega_n^2[{}^L\mathbf{N}'_W(0;\omega_n)] \\ EI[{}^L\mathbf{N}''_W(0;\omega_n)] \end{bmatrix}. \tag{35}$$

3.2.2. Spectral element for the right semi-infinite beam

By following the same procedure as used for the left semi-infinite moving beam, we can derive the spectral components of the displacement fields within the right semi-infinite moving beam shown in Fig. 4(b) as follows:

$$\begin{aligned} {}^RU_n(x) &= [{}^R\mathbf{N}_U(x;\omega_n)]\{{}^R\mathbf{d}_n\}, \\ {}^RW_n(x) &= [{}^R\mathbf{N}_W(x;\omega_n)]\{{}^R\mathbf{d}_n\}, \end{aligned} \tag{36}$$

where

$$\begin{aligned} [{}^R\mathbf{N}_U(x;\omega_n)] &= [{}^R\mathbf{E}_U(x;\omega_n)][{}^R\mathbf{X}_n]^{-1}, \\ [{}^R\mathbf{N}_W(x;\omega_n)] &= [{}^R\mathbf{E}_W(x;\omega_n)][{}^R\mathbf{X}_n]^{-1}, \end{aligned} \tag{37}$$

with the use of following definitions:

$$\begin{aligned} [{}^R\mathbf{E}_U(x;\omega_n)] &= [\,\mathrm{e}^{\mathrm{i}\kappa_{n2}x} \quad 0 \quad 0\,], \\ [{}^R\mathbf{E}_W(x;\omega_n)] &= [\,0 \quad \mathrm{e}^{\mathrm{i}\lambda_{n2}x} \quad \mathrm{e}^{\mathrm{i}\lambda_{n3}x}\,], \end{aligned}$$

$$[{}^R\mathbf{X}_n] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & \mathrm{i}\lambda_{n2} & \mathrm{i}\lambda_{n3} \end{bmatrix}. \tag{38}$$

As shown in Fig. 4(b), $\{{}^R\mathbf{d}_n\}$ represent the spectral nodal dofs defined at the left end of the right semi-infinite moving beam as

$$\{{}^R\mathbf{d}_n\} = \begin{Bmatrix} U_{n2} \\ W_{n2} \\ \Phi_{n2} \end{Bmatrix} = \begin{Bmatrix} {}^R U_n(0) \\ {}^R W_n(0) \\ {}^R W'_n(0) \end{Bmatrix}. \tag{39}$$

Similarly we define the spectral nodal forces and moments as

$$\{{}^R\mathbf{f}_n\} = \begin{Bmatrix} N_{n2} \\ V_{n2} \\ M_{n2} \end{Bmatrix} = \begin{Bmatrix} -N_n(0) \\ -V_n(0) \\ -M_n(0) \end{Bmatrix}. \tag{40}$$

By substituting Eq. (36) into Eq. (11) and applying the result to Eq. (40), we get the relationship between $\{{}^R\mathbf{d}_n\}$ and $\{{}^R\mathbf{f}_n\}$ as

$$[{}^R\mathbf{S}_n]\{{}^R\mathbf{d}_n\} = \{{}^R\mathbf{f}_n\}, \tag{41}$$

where $[{}^L\mathbf{S}_n]$ is the spectral element matrix for the right semi-infinite moving beam given by

$$[{}^R\mathbf{S}_n] = \begin{bmatrix} -EA[{}^R N'_U(0;\omega_n)] \\ EI[{}^R N_W(0;\omega_n)] + \rho I \omega_n^2 [{}^R N'_W(0;\omega_n)] \\ -EI[{}^R N''_W(0;\omega_n)] \end{bmatrix}. \tag{42}$$

4. Assembly of the spectral elements

The spectral element matrices can be assembled in a completely analogous way to that used in the classic FEM [37]. Though only one spectral element is enough to get exact solution for the whole finite beam without any discontinuities throughout the spatial domain of problem in terms of the material properties, geometry, or external loads, we assume that an axial moving uniform beam is divided into total $N+2$ finite elements including two semi-infinite beam parts, where N is the number of finite elements for the finite span of beam between two simple supports (i.e., within the spatial domain of problem). The spectral element matrices $[{}^L\mathbf{S}_n]$ and $[{}^R\mathbf{S}_n]$ for the one-noded left and right semi-infinite beams are the 3×3 matrices and the spectral element matrix $[{}^n\mathbf{S}_n]$ for the nth finite element within the finite span of beam is the 6×6 matrix. The left semi-infinite beam is first assembled with the farmost left finite element (the first finite element) within the finite span of beam and the right semi-infinite beam with the farmost right finite element (the Nth finite element) as follows:

$$\begin{aligned} [{}^{L+1}\mathbf{S}_n] &= [{}^L\mathbf{S}_n] \oplus [{}^1\mathbf{S}_n], \\ [{}^{N+R}\mathbf{S}_n] &= [{}^N\mathbf{S}_n] \oplus [{}^R\mathbf{S}_n], \end{aligned} \tag{43}$$

where the symbol $\oplus$ denotes the finite elements assembly procedure. The (3×3) matrix $[{}^L\mathbf{S}_n]$ has the (3×1) nodal dofs vector $\{{}^L\mathbf{d}_n\}$, while the (6×6) matrix $[{}^1\mathbf{S}_n]$ has the (6×1) nodal dofs vector $\{\mathbf{d}_n\}$ (see Eq. 15). The first three components $\{U_{n1}, W_{n1}, \Phi_{n1}\}$ of $\{\mathbf{d}_n\}$ are same as $\{{}^L\mathbf{d}_n\}$ due to the connectivity at the connection between the left-semi infinite beam and the first finite element of the finite span of beam. So, for the assembly represented by the first equation of (39), the matrix $[{}^L\mathbf{S}_n]$ will be simply summed with the $(1,1)$ submatix of $[{}^1\mathbf{S}_n]$, after partitioning $[{}^L\mathbf{S}_n]$ into (2×2) submatrices. In this way, the effects of the left (or right) semi-infinite beam are coupled to the finite span of beam between two simple supports.

In a similar way, we obtain the global dynamic stiffness matrix for the whole axially moving beam:

$$[\mathbf{S}_n^G] = [^{L+1}\mathbf{S}_n] \oplus [^2\mathbf{S}_n] \oplus \cdots [^n\mathbf{S}_n] \cdots \oplus [^{N-1}\mathbf{S}_n] \oplus [^{N+R}\mathbf{S}_n] \tag{44}$$

and the associated global spectral nodal dofs vector $\{\mathbf{d}_n^G\}$ and the global spectral forces vector $\{\mathbf{f}_n^G\}$ to derive a global system dynamic equation, after applying the related boundary conditions, in the form as

$$[\mathbf{S}_n^G(\omega_n)]\{\mathbf{d}_n^G\} = \{\mathbf{f}_n^G\} \tag{45}$$

The eigenfrequencies ω_{NAT} can be obtained by searching the frequencies ω_n at which the determinant of the global dynamic stiffness matrix vanishes as follows:

$$\det[\mathbf{S}_n^G(\omega_{\mathrm{NAT}})] = 0. \tag{46}$$

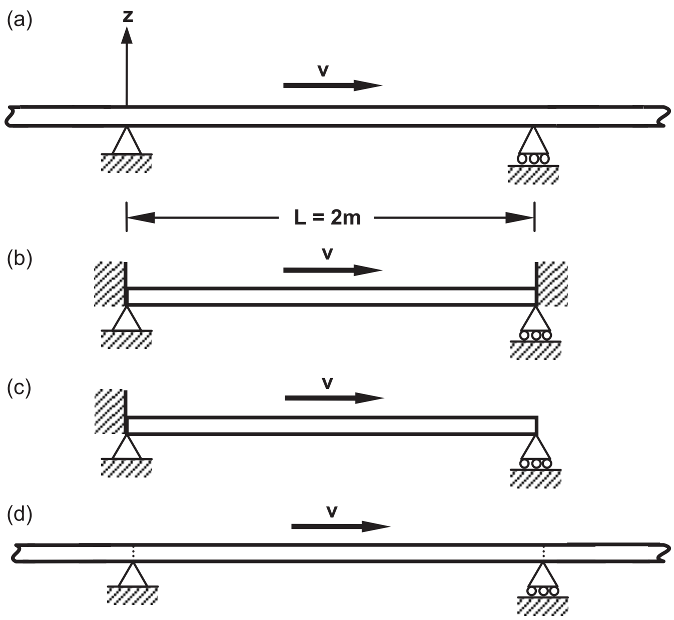

Fig. 5. Three different treatments of the axial boundary conditions at two simple supports: (a) original problem, (b) the fixed–fixed finite beam treatment, (c) the fixed–free finite beam treatment, and (d) the infinite beam treatment by using the semi-infinite spectral elements.

Table 1
Eigenfrequencies (Hz) of an axially moving uniform beam vs. its axial speed for three different treatments of axial boundary conditions

Axial speed (m/s)	Treatments of axial boundary conditions	Bending modes				Axial mode
		1st	2nd	3rd	4th	1st
0	Fixed–free finite beam	2.91	11.64	26.19	46.55	641.70
	Fixed–fixed finite beam	2.91	11.64	26.19	46.55	654.32
	Infinite beam	4.54 + 0.82i	14.69 + 1.44i	30.70 + 2.09i	52.52 + 2.73i	–
4	Fixed–free finite beam	2.70	11.50	26.07	46.44	641.47
	Fixed–fixed finite beam	2.70	11.50	26.07	46.44	654.23
	Infinite beam	4.36 + 0.85i	14.57 + 1.46i	30.59 + 2.10i	52.42 + 2.74i	–
8	Fixed–free finite beam	2.01	11.07	25.70	46.10	641.14
	Fixed–fixed finite beam	2.01	11.07	25.70	46.10	654.95
	Infinite beam	3.83 + 0.95i	14.21 + 1.50i	30.25 + 2.12i	52.09 + 2.76i	–
11.64	Fixed–free finite beam	0.00	10.40	25.14	45.59	641.73
	Fixed–fixed finite beam	0.00	10.40	25.14	45.59	653.53
	Infinite beam	3.03 + 1.20i	13.65 + 1.58i	29.75 + 2.17i	51.63 + 2.79i	–

5. Numerical results and discussions

A uniform beam which is moving at the axial speed of v m/s over two simple supports of distance $a = 2$ m is considered as an illustrative example (Fig. 5(a)). The beam has the thickness $h = 5$ mm, width $b = 0.5$ m, Young's modulus $E = 73$ GPa, and the mass density $\rho = 2770\,\text{kg/m}^3$.

Table 1 compares the eigenfrequencies of the beam at four axial speeds, for three different treatments for the boundary conditions of the finite beam part between two simple supports shown in Fig. 5. The first treatment, which is denoted by 'fixed–fixed finite beam' in the table and figures through out the paper, is to completely ignore the semi-infinite beam parts outside of the two simple supports to consider only the finite beam part by simply applying the fixed axial boundary condition at two simple supports as shown in Fig. 5(b). The second

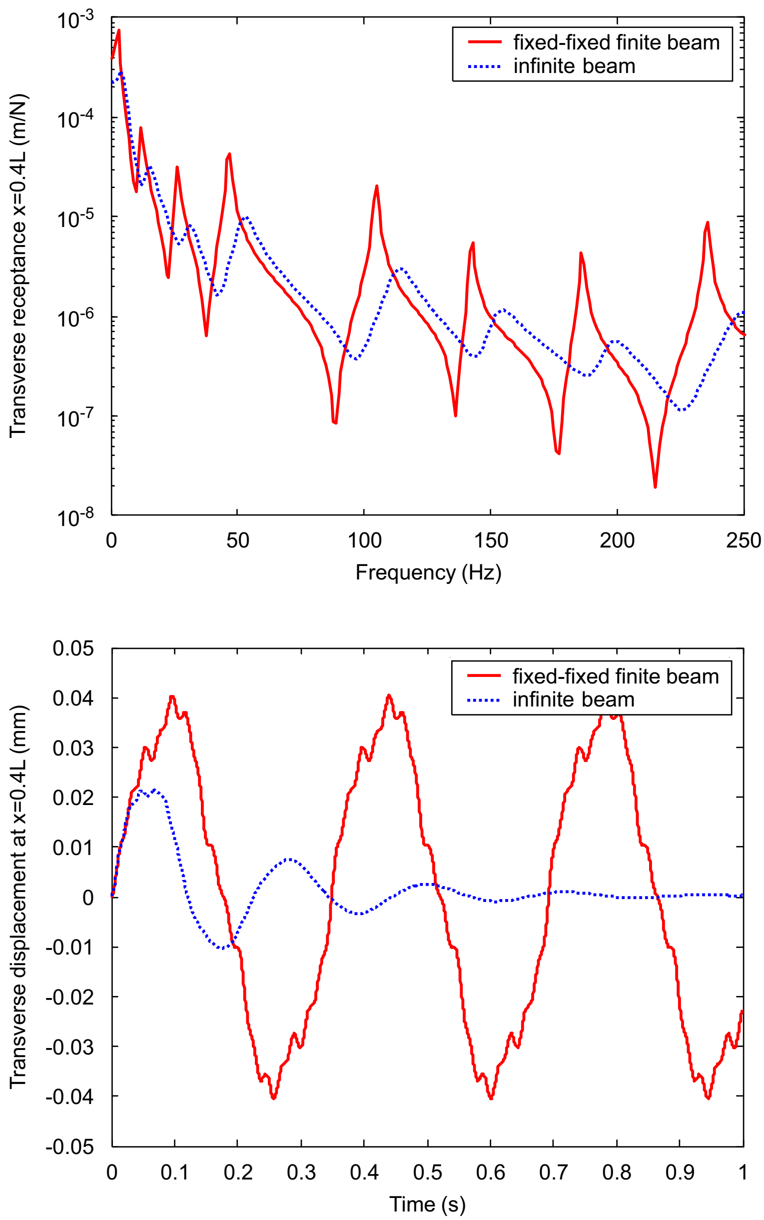

Fig. 6. The frequency response function and the time history for the transverse vibration at $x = 0.4L$ (when the axial speed is 0 m/s), for the fixed–fixed finite beam treatment and the infinite beam treatment.

treatment denoted by 'fixed–free finite beam' is the same as the first treatment except for applying the fixed axial boundary condition at one simple support and the free axial boundary condition at the other simple support as shown in Fig. 5(c). The last treatment denoted by 'infinite beam' is to consider the axially moving beam as an infinite beam by taking into accounts both the left and the right semi-infinite beam parts as the semi-infinite spectral beam elements as shown in see Fig. 5(d). One should notice that the simply supported boundary conditions are applied at two simple supports for the transverse vibration, for all different treatments of the semi-infinite beam parts. From Table 1, we observe the following:

(a) The semi-infinite beam parts certainly play a role to increase the natural frequencies (the real parts of eigenfrequencies) of the bending vibration providing 'damping' (the imaginary parts of eigenfrequencies).

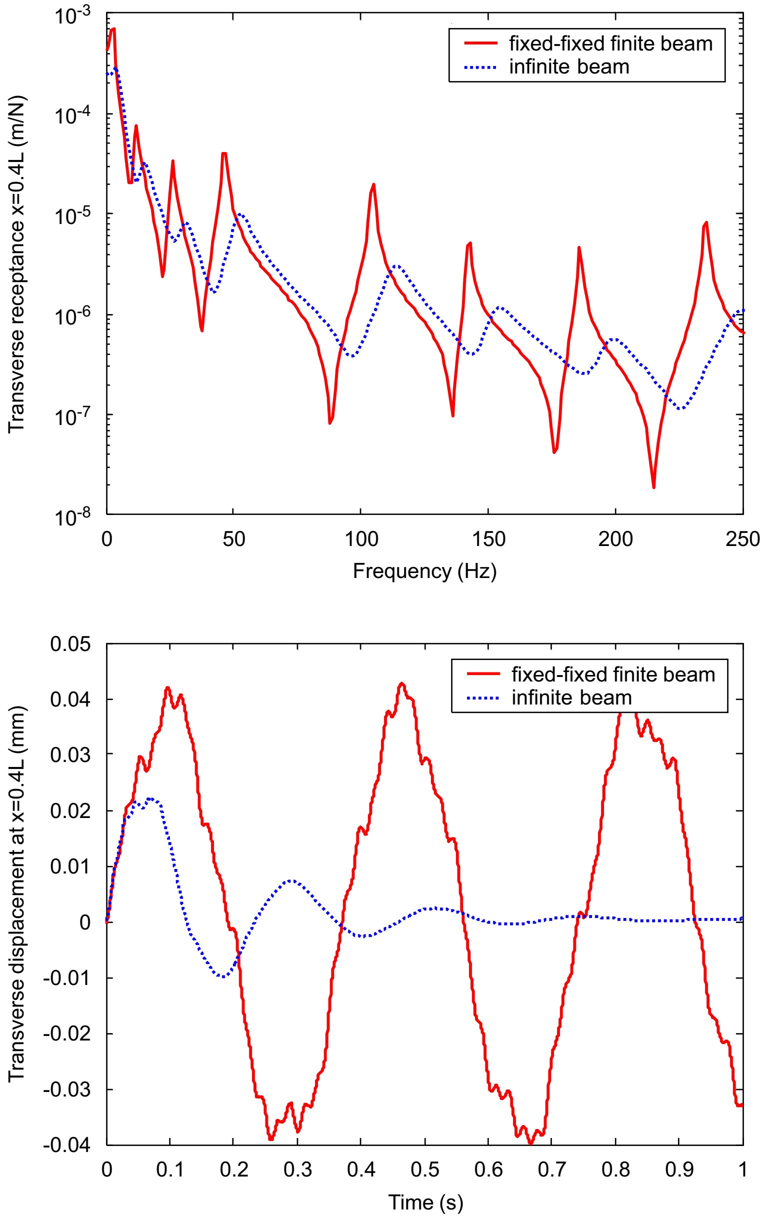

Fig. 7. The frequency response function and the time history for the transverse vibration at $x = 0.4L$ (when the axial speed is 4 m/s), for the fixed–fixed finite beam treatment and the infinite beam treatment.

Physically this seems to be true because the semi-infinite beam parts should increase the bending rigidity of the finite beam between two simple supports and the vibration energy will be dissipated away through the semi-infinite beam parts which play as the energy absorbing media.

(b) The fixed–free finite beam or fixed–fixed finite beam treatments seems to mislead us as if the axial vibration modes exist in the really infinite beam, by providing the corresponding natural frequencies as displayed in the last column of Table 1. This cannot be true from a physical point of view because there are no backward (returning) waves within a really infinite beam to form the standing waves, i.e., axial vibration modes. This misleading problem can be resolved by treating both semi-infinite beam parts as the semi-infinite spectral beam models.

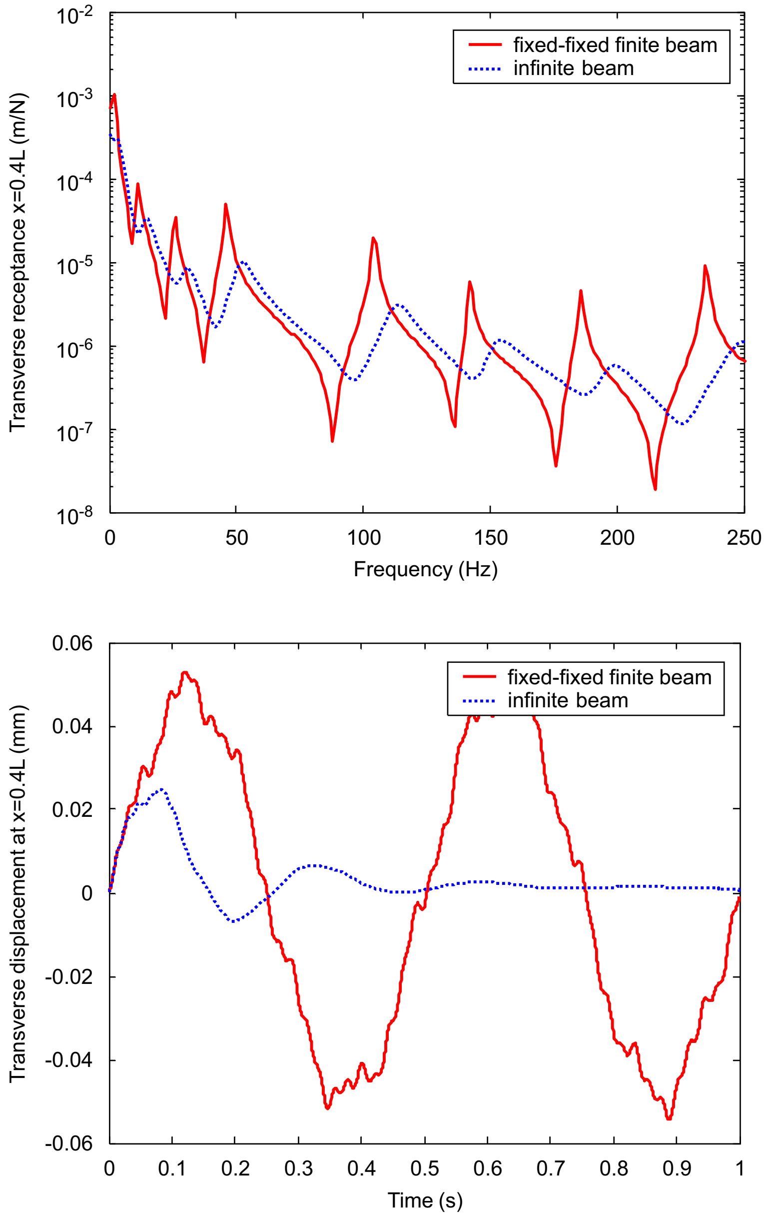

Fig. 8. The frequency response function and the time history for the transverse vibration at $x = 0.4L$ (when the axial speed is 8 m/s), for the fixed–fixed finite beam treatment and the infinite beam treatment.

(c) As reported by many researchers (e.g., Refs. [13,30–32]), Table 1 also shows that the natural frequencies of the beam decreases gradually as its axial speed is increased and finally vanishes at a certain critical axial speed to result in the divergence instability. One significant observation from Table 1 is that the critical speed is significantly underestimated by completely ignoring the left and right semi-infinite beam parts by applying the fixed–free or fixed–fixed axial boundary at two simple supports.

Figs. 6–8 display the frequency-domain and time-domain transverse responses measured at $x = 0.4L$ for three different axial speeds 0, 4, and 8 m/s, respectively, for the cases of the fixed–fixed finite beam treatment and the infinite beam treatment. For the case of the fixed–fixed finite beam treatment, it is obvious from Figs. 6–8 that the dynamic responses at all axial speeds are periodic without damping and the fundamental frequency is getting smaller as the axial speed is increased. For the case of the infinite beam treatment, Figs. 6–8 show that the fundamental frequency is also getting smaller as the axial speed is increased, but the dynamic responses are damped out with time.

From the observations from Table 1 and Figs. 6–8, we may conclude that it is very important to treat the semi-infinite beam parts outside of two simple supports in a very proper way to predict accurate, physically reasonable dynamic characteristics of axially moving beams. When compared with the fixed–free finite beam and the fixed–fixed finite beam treatments, the present infinite beam treatment by applying the semi-infinite spectral beam models for both semi-infinite beam parts is found to provide physically more realistic results.

6. Conclusions

For accurate prediction of the dynamic characteristics and stability for an axially moving beam, it will be important to treat its boundary conditions and the continuously incoming and outgoing semi-infinite beam parts outside of the finite span of beam over two simple supports in an accurate way. From the physical point of view, completely ignoring the effects of the continuously incoming and outgoing semi-infinite beam parts and applying fixed boundary conditions for the longitudinal vibration are not so realistic. Thus, this paper investigates the effects of the continuously incoming and outgoing semi-infinite beam parts on the dynamic characteristics and stability of an axially moving beam by using the SEM.

It is numerically shown that the continuously incoming and outgoing semi-infinite beam parts have significant influence on the dynamics and stability of the axially moving beam. For the case of the fixed–fixed finite beam treatment, it is shown that the dynamic responses at all axial speeds are periodic without damping and the fundamental frequency is getting smaller as the axial speed is increased. For the case of the infinite beam treatment, however, it is shown that the fundamental frequency is also getting smaller as the axial speed is increased, but the dynamic responses are damped out with time. Based on these numerical investigations, we conclude that the effects of incoming and outgoing semi-infinite beam parts should be taken into account in the analysis to acquire improved dynamic characteristics and stability for such axially moving beams.

Though the effects of wave propagations through the incoming and outgoing semi-infinite beam parts (by using the semi-infinite spectral elements) as well as the effect of the mass transport across the boundaries (by applying the Hamilton's principle extended for the systems of changing mass) have been taken into account in this study, detailed aspects of energy flux through the boundaries should be further investigated by both theory and experiment.

Acknowledgment

This work was supported by Inha University Research Grant.

Appendix A

The spectral element matrix $[\mathbf{S}_n]$ is given by

$$[\mathbf{S}_{Un}] = [\mathbf{X}_n^{-1}]^{\mathrm{T}}([\mathbf{R}_{Un}] + [\mathbf{R}_{Wn}])[\mathbf{X}_n^{-1}] \tag{A.1}$$

where

$$[\mathbf{R}_{Un}] = \begin{bmatrix} Y_{n11} & 0 & 0 & Y_{n14} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ Y_{n14} & 0 & 0 & Y_{n44} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{A.2}$$

$$Y_{nij} = \frac{e^{i(k_{ni}+k_{nj})l} - 1}{k_{ni} + k_{nj}} (iEAk_{ni}k_{nj} - \rho A\omega_n^2), \tag{A.3}$$

$$[\mathbf{R}_{Wn}] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & X_{n11} & X_{n12} & 0 & X_{n13} & X_{n14} \\ 0 & X_{n12} & X_{n22} & 0 & X_{n23} & X_{n24} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & X_{n13} & X_{n23} & 0 & X_{n33} & X_{n34} \\ 0 & X_{n14} & X_{n24} & 0 & X_{n34} & X_{n44} \end{bmatrix}, \tag{A.4}$$

$$X_{nij} = \frac{e^{i(\lambda_{ni}+\lambda_{nj})l} - 1}{\lambda_{ni} + \lambda_{nj}} \{-iEI\lambda_{ni}^2\lambda_{nj}^2 - iR_n\lambda_{ni}\lambda_{nj} - \rho Av\omega_n(\lambda_{ni} - \lambda_{nj}) - \rho A\omega_n^2\} \tag{A.5}$$

and the matrix $[\mathbf{X}_n]$ is given by Eq. (19) in the main text.

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